Graphs of quadratic equations

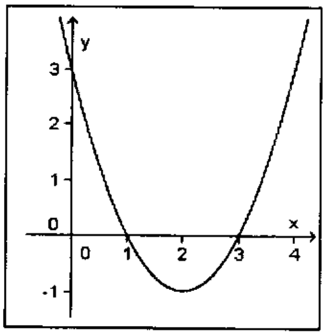
What you should know
How to complete the square on a quadratic expression.
How to use Venn diagrams.

New idea
The graph of $y = (x + a)^2 + b$ has a minimum at $(-a, b)$.

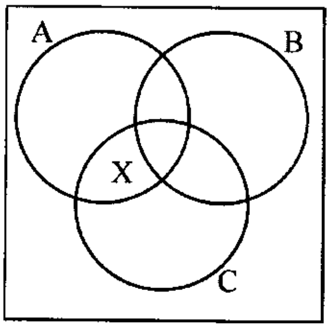
Task: Categorising quadratic curves

Think about all quadratic curves with equation $y = (x + a)^2 + b$.
Think also about these three properties.

A: The turning point has a positive x-value.
B: The turning point has a positive y-value.
C: The y-intercept is positive.



The quadratic curve shown on the right satisfies properties A and C but not B.
Its equation can be written as $y = (x - 2)^2 - 1$.
You could write this in the region marked with an X in the Venn diagram below.



- Copy the Venn diagram.
- Can you find a quadratic which doesn't satisfy any of the properties? Write this in the region outside the three circles.
- Can you find one equation for each of the other six regions?
- Is it possible to find an equation for every region?

Take it further
Find three other properties A, B and C for which all eight regions can be filled in.

Where this goes next
At A level you will learn more about the usefulness of completing the square in Core Mathematics. You will also learn another method for finding maximum and minimum points on cubics and other curves.

Interpreting graphs

What you should know

A graph of the form $y = mx + c$ represents a straight line where m is the gradient of the line and c is the value of the y-intercept.
How to draw and interpret velocity–time graphs.

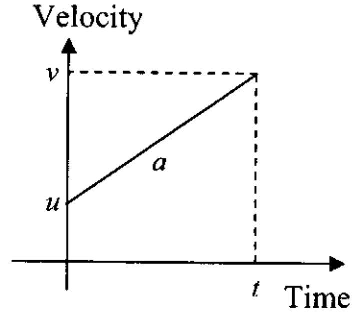
New ideas

A simple equation for an object travelling with constant acceleration is $v = u + at$, where u is its initial velocity (speed in a given direction), v is its final velocity, a is its acceleration and t is the time from the beginning to the end of its motion.
The area under a velocity–time graph represents the displacement (distance travelled in a given direction).

Task: Velocity–time graphs

A car accelerates from 10 ms^{-1} at a constant rate when leaving a built-up area.
It takes 6 seconds to reach a velocity of 22 ms^{-1} (about 50 miles per hour).

- Draw a velocity–time graph to represent this motion.
- The graph should be a straight line. You already know the equation $y = mx + c$ to describe a straight line. If you use $v = u + at$ to describe the motion, what properties would v, t, u and a represent on the graph?
- Use the equation $v = u + at$ to find the car's acceleration.
  What is the gradient of the line you have drawn?
- The area under a velocity–time graph represents displacement.
  The letter s is used for displacement.
  Find the displacement, s, of the car for this journey.
- The diagram shows the general graph for motion with constant velocity. Find a formula for the displacement, s, in terms of u, v and t.
  Check that this formula works for the example above.



Take it further

- Try substituting $v = u + at$ into your formula for s and simplifying it.
  This should give you a formula for s, in terms of u, a and t.
  Check that this formula works for the example above, too.
- A dropped object falls to Earth with an acceleration of approximately 10 ms^{-2} and has initial velocity, u, of 0.
  Use the formulae you found to work out how far it would fall and what its velocity would be after 1 second, 2 seconds, 5 seconds, 10 seconds, etc.
  What is the problem with this model?

Where this goes next

At A level constant acceleration formulae are studied in Mechanics.

Solving quadratic equations

What you should know

How to solve a quadratic equation $ax^2 + bx + c = 0$ using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

New idea

When the number under the square root is negative you usually stop because "you can't square root a negative number" … but what if you could?

If you wanted to find a number that was the square root of -1 it couldn't be positive, negative or zero. (Why not?) In fact it can't be any 'real' number, so it must be an 'imaginary' number. The square root of -1 is called i.

So $\sqrt{-1} = i$ and $i^2 = -1$

Task: Complex numbers

If you solve the quadratic equation

$$x^2 - 4x + 5 = 0$$

using the formula, you get

$$x = \frac{4 \pm \sqrt{-4}}{2}$$

so one of the roots is

$$x = \frac{4 + \sqrt{4}\sqrt{-1}}{2}$$

$$= \frac{4 + 2i}{2} = 2 + i$$

Check:

$(2 + i)^2 - 4(2 + i) + 5$

$= 4 + 2i + 2i - 1 - 8 - 4i + 5$

$= 0$

This value of x works in the equation!

Numbers like $2 + i$ that have a **real** and an **imaginary** part are called **complex numbers**.

- Why does $(2 + i)^2$ expand to $4 + 2i + 2i - 1$?
- What is the other root of $x^2 - 4x + 5 = 0$?
  Does this root work in the equation?
- Can you find other quadratic equations that have a negative number under the square root?
  Can you solve these equations?

Take it further

- What do the graphs of this type of quadratic equation look like?
- Find out more about complex numbers.

Where this goes next

At A level complex numbers are studied in Further Mathematics.

Plotting curves

What you should know

How to plot a curve when given y as a function of x by making a table of values.

New idea

You can plot a curve of y against x when both x and y are defined in terms of a third variable such as t. This type of variable is often known as a parameter.

Task: Parametric equations

- Copy and complete this table of values for $y = 3x - \frac{1}{20}x^2$.

x	0	10	20	30	40	50	60
$3x$		30					
$-\frac{1}{20}x^2$		−5					
y		25					

- Draw the graph using the same scales for x and y.
  You are going to use it later so make sure you keep it.
- The graph looks like the path of a football after a goal kick.
  Think of two other things that it looks like.

In the case of the goal kick, the distances x and y could be in metres and the points given every 1 second.
Call t the time in seconds from the moment when the goalkeeper kicks the ball.
Here is the table of values for x in terms of t.

t	0	1	2	3	4	5	6
x	0	10	20	30	40	50	60

- Write down the formula for x in terms of t.
- Now write out the table of values for y in terms of t.
  You can use the values of x to find the values of t and then use the first table to find the values of y. For example, when $t = 1$, $x = 10$ and $y = 25$.
- The formula for y in terms of t has the form $y = at - bt^2$ where a and b are numbers. Find the values of a and b.

Above you met a simple model for the path of a football.
A more refined model is given by the *parametric* equations $x = 10t - \frac{1}{5}t^2$, $y = 30t - 5t^2$.

- Make tables of values for x and y for $0 \le t \le 6$ and plot the graph of y against x on the same piece of graph paper as you used before.
- What do you think the refined model has allowed for?

Take it further

Investigate Lissajous curves. (See investigation on separate sheet.)

Where this goes next

At A level parametric equations are studied in Core Mathematics and Mechanics.